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CCC FORCING AND SPLITTING REALS

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ABSTRACT

The results of this paper were motivated by a problem of Prikry who asked if it is relatively consistent with the usual axioms of set theory that every nontrivial ccc forcing adds a Cohen or a random real. A natural dividing line is into weakly distributive posets and those which add an unbounded real. In this paper I show that it is relatively consistent that every nonatomic weakly distributive ccc complete Boolean algebra is a Maharam algebra, i.e. carries a continuous strictly positive submeasure. This is deduced from the P-ideal dichotomy, a statement which was first formulated by Abraham and Todorcevic [AT] and later extended by Todorcevic [T]. As an immediate consequence of this and the proof of the consistency of the P-ideal dichotomy we obtain a ZFC result which says that every absolutely ccc weakly distributive complete Boolean algebra is a Maharam algebra. Using a previous theorem of Shelah [Sh1] it also follows that a modified Prikry conjecture holds in the context of Souslin forcing notions, i.e. every nonatomic ccc Souslin forcing either adds a Cohen real or its regular open algebra is a Maharam algebra. Finally, I also show that every nonatomic Maharam algebra adds a splitting real, i.e. a set of integers which neither contains nor is disjoint from an infinite set of integers in the ground model. It follows from the result of [AT] that it is consistent relative to the consistency of ZFC alone that every nonatomic weakly distributive ccc forcing adds a splitting real.

1. Introduction

Given two forcing notions $\mathcal{P}$ and $\mathcal{Q}$, let us write $\mathcal{P} \leq \mathcal{Q}$ iff forcing with $\mathcal{Q}$ introduces a $\mathcal{P}$-generic over V. It is fairly easy to see that this is equivalent to saying that there is $p \in \mathcal{P}$ and an embedding of the complete Boolean algebra

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$RO(\mathcal{P} \restriction p)$ into $RO(\mathcal{Q})$. Let Σ be a given class of posets. We say that $\Sigma_0 \subseteq \Sigma$ is a **basis** for Σ if for every $\mathcal{Q} \in \Sigma$ there is $\mathcal{P} \in \Sigma_0$ such that $\mathcal{P} \leq \mathcal{Q}$. We are interested in finding a basis for the class of ccc forcing notions. Clearly, both Cohen forcing $\mathcal{C}$ and random real forcing $\mathcal{R}$ have to be in any such basis. Prikry asked if it is consistent that $\{\mathcal{C}, \mathcal{R}\}$ form a basis for the class of all ccc posets. This is equivalent to saying that every nontrivial ccc poset adds a Cohen or a random real. Another version of this problem is to identify a basis for the class of appropriately definable ccc posets. Under ZFC definable is interpreted to mean the class of Souslin posets, i.e. posets $\mathcal{P}$ such that the domain of $\mathcal{P}$ is an analytic set of reals and both the order and the incompatibility relation of $\mathcal{P}$ are analytic. Under suitable large cardinal or determinacy assumptions we can replace Souslin by some higher order definability. One common property of both Cohen and random reals is that they are both splitting reals, i.e. they neither contain nor are disjoint from an infinite set of integers in the ground model. Thus, one test question going in the direction of Prikry's conjecture is whether it is relatively consistent that every nontrivial ccc forcing adds a splitting real.

Another related problem is a well-known question of von Neumann [Mau] who asked if every weakly distributive complete ccc Boolean algebra is a measure algebra. Recall that a poset $\mathcal{P}$ is **weakly distributive** iff every real (i.e. element of ω^ω) in $V^{\mathcal{P}}$ is dominated by a ground model real, i.e. $\mathcal{P}$ is ω^ω-bounding. Maharam [Mah] formulated the notion of a continuous submeasure and found an algebraic characterization for a complete Boolean algebra to carry one; such algebras are now known as Maharam algebras. Any Maharam algebra is both ccc and weakly distributive and it is not known if it is necessarily a measure algebra. The question whether every Maharam algebra is, in fact, a measure algebra is the well known Control Measure Problem.

In this paper I show that the P-ideal dichotomy, formulated by Abraham and Todorcevic [AT] and later extended by Todorcevic [T], implies that every weakly distributive ccc complete Boolean algebra carries a positive continuous submeasure, i.e. is a Maharam algebra. The P-ideal dichotomy is a consequence of the Proper Forcing Axiom and it was shown in [T] that it is consistent with GCH assuming the existence of a supercompact cardinal. As an immediate consequence of our result and the proof of the consistency of the P-ideal dichotomy we obtain that in ZFC every absolutely ccc weakly distributive complete Boolean algebra is a Maharam algebra. Moreover, using a previous result of Shelah ([Sh1], see also [Vel]) we deduce that a modified version of Prikry's conjecture holds in the context of Souslin ccc forcing: i.e. every Souslin ccc forcing adds a Cohen real or is a Maharam algebra. Indeed, Shelah showed that any Souslin ccc forcing which is nowhere weakly distributive adds a Cohen real. It is not known if it is consistent that this holds without any definability restriction on the forcing. However, Błaszyck and Shelah [BłSh], using a previous result of Shelah [Sh2], showed that it is relatively consistent with ZFC that every nonatomic σ-centered forcing notion adds a Cohen real.

Finally, I show that every nonatomic Maharam algebra adds a splitting real. Using the fact that every countably generated ccc complete Boolean algebra is of size at most continuum and the original result from [AT] it follows that it is consistent, relative to the consistency of the ZFC alone, that every nonatomic weakly distributive ccc forcing adds a splitting real.

The main motivation for our proof comes from the PhD thesis of Quickert [Qu1] who introduced some of the key concepts used in this paper and showed that under the P-ideal dichotomy no nontrivial weakly distributive ccc forcing has the Sacks property*.

2. P-ideal dichotomy and continuous submeasures

Recall that a submeasure on a Boolean algebra $\mathcal{B}$ is a function $\mu\colon \mathcal{B} \to [0,1]$ such that $\mu(a) = 0$ iff $a = \mathbf{0}$, $a \leq b$ implies $\mu(a) \leq \mu(b)$, and $\mu(a \vee b) \leq \mu(a) + \mu(b)$, for every $a, b \in \mathcal{B}$. We say that μ is **exhaustive** if for every sequence $\{a_n\colon n < \omega\}$ of disjoint elements of $\mathcal{B}$ we have $\lim_{n\to\infty} \mu(a_n) = 0$, and that it is **uniformly exhaustive** if for every $\epsilon > 0$ there is an integer n such that any family of pairwise disjoint elements of $\mathcal{B}$ each of μ-submeasure at least ϵ has size at most n. Finally μ is called **continuous** if for every sequence $\{a_n\colon n < \omega\}$ of elements of $\mathcal{B}$ if $\limsup_n a_n = \liminf_n a_n = a$, for some a, then $\lim_n \mu(a_n) = \mu(a)$. It was shown by Maharam ([Mah], Lemma 1) that if $\mathcal{B}$ is a complete Boolean algebra and μ is a submeasure on $\mathcal{B}$ such that $\lim_n \mu(a_n) = 0$, for every decreasing sequence $\{a_n\}_n$ with $\bigwedge_n a_n = \mathbf{0}$, then μ is a continuous submeasure. We call complete Boolean algebras carrying such submeasures **Maharam algebras**. One formulation of the famous Control Measure Problem asks if every Maharam algebra is, in fact, a measure algebra.

Our goal in this section is to show that it is relatively consistent, modulo a supercompact cardinal, to assume that every weakly distributive ccc complete Boolean algebra is a Maharam algebra. In fact, we will deduce this from the

* Recall that a forcing notion $\mathcal{P}$ has the **Sacks property** if for every function $f \in \omega^\omega$ in a generic extension $V[G]$ by the forcing $\mathcal{P}$ there is a sequence $\langle I_n\colon n < \omega\rangle$ in the ground model such that $|I_n| \leq n$, for all n, and $f \in \prod_n I_n$.

P-ideal dichotomy. Recall that an ideal $\mathcal{I}$ of subsets of a set X is called a **P-ideal** if for every sequence $\{A_n: n < \omega\}$ of elements of $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subseteq_* A$, for all n. Here $\subseteq_*$ denotes inclusion modulo finite sets. We say that a set Y is **orthogonal** to a family $\mathcal{A}$ provided the intersection of Y with any element of $\mathcal{A}$ is finite.

Let $(*)_\kappa$ be the following principle:

$(*)_\kappa$ *Let X be a set of size at most κ and let $\mathcal{I}$ be a P-ideal of countable subsets of X. Then one of the following two alternatives holds:*

(a) *there is an uncountable subset Y of X such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$;*

(b) *we can write $X = \bigcup_n X_n$, where X_n is orthogonal to $\mathcal{I}$, for each n.*

The **P-ideal dichotomy** is the statement that $(*)_\kappa$ holds, for all cardinals κ. This principle, which follows from the Proper Forcing Axiom, was first studied by Abraham and Todorcevic [AT] who proved that $(*)_{2^{\aleph_0}}$ is relatively consistent with CH and used it to show that some consequences of Martin's Axiom and the negation of the Continuum Hypothesis are relatively consistent with CH. Later, Todorcevic [T] extended this result by showing that the full version of the P-ideal dichotomy is relatively consistent with GCH assuming the existence of a supercompact cardinal.

Let $\mathcal{B}$ be a complete ccc weakly distributive Boolean algebra. Following Quickert let us define the following ideal $\mathcal{I}$ on $[\mathcal{B} \setminus \{\mathbf{0}\}]^{\leq\omega}$:

$$X \in \mathcal{I} \text{ if and only if } \Vdash_{\mathcal{B}} X \cap \dot{G} \text{ is finite.}$$

Thus, $X \in \mathcal{I}$ iff there is a maximal antichain $\mathcal{A}$ such that every member of $\mathcal{A}$ is compatible with at most finitely many members of X. The following two lemmas are from [Qu2]. We reproduce the proofs for completeness.

LEMMA 1: *$\mathcal{I}$ is a P-ideal.*

Proof: Suppose $X_n \in \mathcal{I}$, for all n. We may assume all the X_n are infinite and fix, for each n, an enumeration $X_n = \{p_{n,k}: k < \omega\}$. Define a name $\dot{f}$ for a function in ω^ω by letting $\dot{f}(n)$ be the name for the least integer k such that $p_{n,l} \notin \dot{G}$, for all $l \geq k$. By the definition of $\mathcal{I}$ this is well defined. Now, by weak distributivity and the ccc of $\mathcal{B}$ we can fix a function $g \in \omega^\omega$ such that $\Vdash \dot{f} \leq_* g$. Let

$$X = \bigcup_n X_n \setminus \{p_{n,i}: i < g(n)\}.$$

It follows that $\Vdash X \cap \dot{G}$ is finite, i.e. $X \in \mathcal{I}$ and clearly we have $X_n \subseteq_* X$, for all n. ∎

LEMMA 2: *There is no uncountable X such that $[X]^{\leq\omega} \subseteq \mathcal{I}$.*

Proof: Assume otherwise and let X be a counterexample. Since $\mathcal{B}$ satisfies the ccc there is a condition $b \in \mathcal{B}$ such that $b \Vdash$ "$X \cap \dot{G}$ is uncountable". Then, again by the ccc, there is a countable $A \subseteq X$ such that $b \Vdash$ "$A \cap \dot{G}$ is infinite". Therefore, $A \notin \mathcal{I}$, a contradiction. ∎

Quickert [Qu2] used these two lemmas in her proof that under the P-ideal dichotomy no nonatomic ccc forcing has the Sacks property. Moreover, she showed in [Qu1] that if the P-ideal dichotomy holds then every weakly distributive ccc forcing $\mathcal{P}$ satisfies the σ-finite chain condition, i.e. can be written as $\mathcal{P} = \bigcup_n \mathcal{P}_n$, where $\mathcal{P}_n$ has no infinite subset of pairwise incompatible elements, for each n.

THEOREM 1: *Let $\mathcal{B}$ be a weakly distributive ccc complete Boolean algebra and let $\mathcal{I}$ be the Quickert ideal of $\mathcal{B}$. Assume $\mathcal{B} \setminus \{\mathbf{0}\}$ can be covered by countably many sets orthogonal to $\mathcal{I}$. Then $\mathcal{B}$ is a Maharam algebra.*

Proof: Let us say that a set $U \subseteq \mathcal{B}$ is **large** if it is downward closed and $A \subseteq_* U$, for every $A \in \mathcal{I}$; i.e. if $\mathcal{B} \setminus U$ is orthogonal to $\mathcal{I}$. Note that our definition of largeness is not the same as the one in [Qu2]. In fact, a set U is large in our sense iff the set $\{-b\colon b \in U\}$ is large in the sense of Quickert. While we could have, of course, used her definition, we find this one slightly more convenient.

Now, by our assumption we can write $\mathcal{B} \setminus \{\mathbf{0}\} = \bigcup_n X_n$, where X_n is orthogonal to $\mathcal{I}$, for all n. By replacing X_n by its upward closure we may assume that if $a \in X_n$ and $a \leq b$ then $b \in X_n$. Moreover, by replacing X_n by $\bigcup_{i \leq n} X_i$, we may assume that $X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots$. Let $U_n = \mathcal{B} \setminus X_n$. Then, by definition, each U_n is large, we have $U_0 \supseteq U_1 \supseteq U_2 \supseteq \cdots$ and $\bigcap\{U_n\colon n < \omega\} = \{\mathbf{0}\}$.

Our first goal is to improve this sequence in order to have the additional property that $U_{n+1} \vee U_{n+1} \subseteq U_n$, for every n, where

$$U \vee V = \{u \vee v\colon u \in U \text{ and } v \in V\}.$$

For this, it is clearly sufficient to show that for every large U there is a large V such that $V \vee V \subseteq U$.

Remark 1: Note that if we assume that $\mathcal{B}$ does not add splitting reals we immediately have that for every n there is k such that $U_k \vee U_k \subseteq U_n$. To see this, assume otherwise and fix for each $k \geq n$, $b_k^0, b_k^1 \in U_k$ such that $b_k = b_k^0 \vee b_k^1 \notin U_n$.

Since each b_k is not in U_n we can show that there is an infinite $I \subseteq \mathbb{N}$ such that $c = \bigwedge\{b_k: k \in I\} \neq \mathbf{0}$. Then we define a name τ for an element of 2^ω by letting $||\tau(k) = 1|| = b_k^1$. Again, by the assumption that no splitting reals are added we find an infinite $J \subseteq I$ and $\epsilon \in \{0,1\}$ such that $d = \bigwedge\{b_k^\epsilon: k \in J\} \neq \mathbf{0}$. Let $l \geq n$ be such that $d \in X_l$. Since X_l is upward closed it follows that $b_k^\epsilon \in X_l$, for every $k \in J$, but if $k \geq l$ we have $b_k^\epsilon \in U_l$, a contradiction.

For a subset V of $\mathcal{B}$ let $V(x) = \{a \in V: x \vee a \in V\}$ and let Z_V be the set of all $x \in V$ such that $V(x)$ is large.

CLAIM 1: *If V is large then so is Z_V.*

Proof: Otherwise there would be an infinite set $X \in \mathcal{I}$ which is disjoint from Z_V. Fix an enumeration $X = \{x_n: n < \omega\}$, and for each n, an infinite set $A_n \in \mathcal{I}$ such that for every $a \in A_n$, $x_n \vee a \notin V$. Since $\mathcal{I}$ is a P-ideal we can find $A \in \mathcal{I}$ such that $A_n \subseteq_* A$, for all n. Now, choose a $1-1$ sequence $\{a_n: n < \omega\}$ of elements of A such that $x_n \vee a_n \notin V$. Since both X and A belong to $\mathcal{I}$ it follows that the set $\{x_n \vee a_n: n < \omega\}$ is also in $\mathcal{I}$. But then there must be an n such that $x_n \vee a_n \in V$, a contradiction. ∎

LEMMA 3: *For every large V there is a large W such that $W \vee W \subseteq V$.*

Proof: Assume otherwise and fix a large V for which this fails. Let $V_0 = V \cap U_0$. Since Z_{V_0} is large by our assumption there are x_0 and y_0 in Z_{V_0} such that $x_0 \vee y_0 \notin V$. Let $V_1 = V_0(x_0) \cap V_0(y_0) \cap U_1$. Since Z_{V_1} is large we can again pick x_1 and y_1 in Z_{V_1} such that $x_1 \vee y_1 \notin V$. Let $V_2 = V_1(x_1) \cap V_1(y_1) \cap U_2$. Then V_2 is large and so is Z_{V_2}. We pick x_2 and y_2 in Z_{V_2} such that $x_2 \vee y_2 \notin V$. We proceed by recursion. Given V_n which is large we can pick x_n and y_n in Z_{V_n} such that $x_n \vee y_n \notin V$. We then let $V_{n+1} = V_n(x_n) \cap V_n(y_n) \cap U_{n+1}$. Notice that for every n and k we have $x_n \vee x_{n+1} \vee x_{n+2} \vee \cdots \vee x_{n+k} \in U_n$.

CLAIM 2: *$\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ belong to $\mathcal{I}$.*

Proof: Since the statement is symmetric let us assume, towards contradiction, that $\{x_n: n < \omega\}$ is not in $\mathcal{I}$. Fix a condition $b \in \mathcal{B} \setminus \{\mathbf{0}\}$ such that

$$b \Vdash \text{“}\{n: x_n \in \dot{G}\} \text{ is infinite”}.$$

Since $\mathcal{B}$ is weakly distributive we can pick a strictly increasing function f in ω^ω and a nonzero $c \leq b$ such that

$$c \Vdash \bigvee\{x_i: f(n) < i \leq f(n+1)\} \in \dot{G}$$

for every n. Let $z_n = \bigvee\{x_i: f(n) < i \leq f(n+1)\}$. Then, by our construction, we have that $z_n \in U_{f(n)}$ and on the other hand $c \leq z_n$, for all n. Since the U_n are downward closed it follows that $c \in \bigcap\{U_l: l < \omega\} = \{\mathbf{0}\}$, a contradiction. ∎

Now, since both $\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ are in $\mathcal{I}$ it follows that $\{x_n \vee y_n: n < \omega\}$ is in $\mathcal{I}$, as well. But this set is disjoint from V and V was supposed to be large, a contradiction. This finishes the proof of Lemma 3. ∎

Now, using Lemma 3 we can improve the original decreasing sequence $U_0 \supseteq U_1 \supseteq \cdots$ to have in addition that $U_{n+1} \vee U_{n+1} \subseteq U_n$, for all n. Let us define a function $\varphi: \mathcal{B} \to [0,1]$ by

$$\varphi(a) = \inf\{2^{-n}: a \in U_n\}.$$

Now we define a submeasure $\mu: \mathcal{B} \to [0,1]$ as follows:

$$\mu(b) = \inf\{\sum_{i=1}^{l} \varphi(a_i): b \leq \bigvee_{i=1}^{l} a_i\} \cup \{1\}.$$

LEMMA 4: *μ is a positive continuous submeasure on $\mathcal{B}$.*

Proof: It is clear that if $a \leq b$ then $\mu(a) \leq \mu(b)$ and that $\mu(a \vee b) \leq \mu(a) + \mu(b)$, for every $a, b \in \mathcal{B}$. We need to show that μ is positive on every nonzero element of $\mathcal{B}$ and that it is continuous. The following fact is immediate.

FACT 1: Suppose $n_1 < n_2 < \cdots < n_k$ and $a_i \in U_{n_i+1}$, for $i = 1, \ldots, k$. Then $\bigvee_{i=1}^{k} a_i \in U_{n_1}$. ∎

From this it follows that if $a \notin U_n$ then $\mu(a) \geq 2^{-n}$, therefore μ is positive. Finally, to see that μ is continuous notice that by Lemma 1 of [Mah] it suffices to prove that if $\{x_n: n < \omega\} \in \mathcal{I}$ then $\lim_n \mu(a_n) = 0$. To show this fix an integer k. Since $\{x_n: n < \omega\} \in \mathcal{I}$ and U_k is large, there is n such that $x_l \in U_k$ for all $l \geq n$. This means that

$$\mu(x_l) \leq \varphi(x_l) \leq 2^{-k}$$

for all $l \geq n$. Since k was abritrary it follows that $\lim_n(a_n) = 0$, as desired. This finishes the proof of Lemma 4 and Theorem 1. ∎

Now using Theorem 1 and Lemma 1 and Lemma 2 we immediately have the following.

COROLLARY 1: *Assume the P-ideal dichotomy. Then every weakly distributive ccc complete Boolean algebra is a Maharam algebra.* ∎

Given a class Σ of forcing notions let us say that a forcing notion $\mathcal{P}$ is Σ-**absolutely ccc** if $\mathcal{P}$ is ccc in $V^{\mathcal{Q}}$, for every $\mathcal{Q}$ in Σ. If Σ is the class of all forcings preserving ω_1 we say simply that $\mathcal{P}$ is **absolutely ccc**. For example, forcing notions which are σ-linked, satisfy the σ-finite chain condition and Souslin forcings are all absolutely ccc. On the other hand, a Souslin tree T is not absolutely ccc for the class of σ-distributive forcings since forcing with T itself destroys the ccc-ness of T.

COROLLARY 2: *Let $\mathcal{B}$ be a weakly distributive complete Boolean algebra which is absolutely ccc for the class of all σ-distributive forcings. Then $\mathcal{B}$ is a Maharam algebra.*

Proof: This is actually a corollary of Theorem 1 and the proof of the consistency of the P-ideal dichotomy from [T]. Namely, what is shown in [T] is the following. Suppose $\mathcal{I}$ is a P-ideal of countable subsets of some set X. Suppose that X is not covered by countable many sets orthogonal to $\mathcal{I}$, but that any subset of X of cardinality smaller than that of X is covered by such a collection. Then there is a forcing notion $\mathcal{P}$ which is $\mathcal{D}$-complete for some σ-complete completeness system $\mathcal{D}$ (in particular is σ-distributive), is α-proper, for all $\alpha < \omega_1$, and adds an uncountable set Y such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$. Now, let $\mathcal{B}$ be a weakly distributive complete Boolean algebra which is absolutely ccc for σ-distributive forcing notions. Let $\mathcal{I}$ be the Quickert ideal of $\mathcal{B}$. If $\mathcal{B}$ is the union of countably many sets orthogonal to $\mathcal{I}$ by Theorem 1 $\mathcal{B}$ is a Maharam algebra. Assume otherwise and let X be a subset of $\mathcal{B}$ of smallest cardinality which is not covered by countably many sets orthogonal to $\mathcal{I}$. Let $\mathcal{P}$ be the poset described above. Then in $V^{\mathcal{P}}$ there would be an uncountable subset Y of X such that $[Y]^{\leq\omega} \subseteq \mathcal{I}$, but this contradicts Lemma 2 above and the fact that $\mathcal{B}$ remains ccc in $V^{\mathcal{P}}$. ∎

COROLLARY 3: *Let $\mathcal{S}$ be a nonatomic Souslin ccc forcing notion. Then either there is $p \in \mathcal{S}$ such that forcing with $\mathcal{S}$ below p adds a Cohen real or else the regular open algebra $RO(\mathcal{S})$ is a Maharam algebra.*

Proof: In [Sh1] Shelah showed that every ccc Souslin forcing which adds an unbounded real adds a Cohen real. Thus if there is a $\mathcal{S}$-name τ for a real and $p \in \mathcal{S}$ such that p forces that τ is not bounded by any ground model real, we are done. Otherwise, we may assume that S is a nonatomic weakly distributive ccc Souslin forcing. Since by [JuSh] the ccc of Souslin posets is absolute, it follows that $RO(S)$ is a Maharam algebra. ∎

3. Splitting reals

A common feature of both Cohen and random reals is that they are splitting reals, i.e. they neither contain nor are disjoint from an infinite set of integers in the ground model. The goal of this section is to prove that under the P-ideal dichotomy every nonatomic ccc weakly distributive forcing adds a splitting real. By Corollary 1 it suffices to show that every nonatomic Maharam algebra adds a splitting real. We start with the following simple lemma.

LEMMA 5: *Let $\mathcal{B}$ be a ccc complete Boolean algebra which does not add splitting reals below any condition. Let X be an infinite subset of $\mathcal{B}$. Then there is an infinite subset Y of X such that either $\bigwedge Y \neq \mathbf{0}$ or $\Vdash_{\mathcal{B}} Y \cap \dot{G}$ is finite.*

Proof: Fix an enumeration $X = \{b_n : n < \omega\}$ and let τ be the name for an element of 2^ω defined by $||\tau(n) = 1|| = b_n$. Since τ is forced not to be a splitting real there is an infinite $I_0 \subseteq \mathbb{N}$ and a nonzero c_0 such that $c_0 \Vdash$ "$\tau \restriction I_0$ is constant." We recursively build an antichain $\{c_\xi : \xi < \delta\}$ and a decreasing mod finite sequence $I_0 \supseteq_* \cdots \supseteq_* I_\xi \supseteq_* \cdots$ such that $c_\xi \Vdash_{\mathcal{B}}$ "$\tau \restriction I_\xi$ is almost constant", for all ξ. At a countable limit stage λ we first diagonalize to find an infinite J such that $J \subseteq_* I_\xi$, for all $\xi < \lambda$. If $\{c_\xi : \xi < \lambda\}$ is not already a maximal antichain, by using the fact that $\tau \restriction J$ is forced not to be a splitting real, we find c_λ incompatible with all the c_ξ, for $\xi < \lambda$, and an infinite $I_\lambda \subset J$ such that $c_\lambda \Vdash_{\mathcal{B}}$ "$\tau \restriction J$ is constant". Since $\mathcal{B}$ is ccc the construction must stop after countably many steps. At this stage we get an infinite I such that $\Vdash$ "$\tau \restriction I$ is almost constant". Let $Y = \{b_n : n \in I\}$. If there is $c \in \mathcal{B} \setminus \{\mathbf{0}\}$ and an integer n such that $c \Vdash_{\mathcal{B}} \tau \restriction (I \setminus n) \equiv 1$, then it follows that $c \leq \bigwedge Y$. Otherwise $\Vdash_{\mathcal{B}}$ "$Y \cap \dot{G}$ is finite ". ∎

THEOREM 2: *Let $\mathcal{B}$ be a nonatomic Maharam algebra. Then forcing with $\mathcal{B}$ adds a splitting real.*

Proof: Let μ be a continuous submeasure on $\mathcal{B}$. If μ is uniformly exhaustive, by a theorem of Kalton and Roberts [KR] $\mathcal{B}$ is a measure algebra and therefore it adjoins a random real. Assume now $\mathcal{B}$ is not uniformly exhaustive and fix an $\epsilon > 0$ which witnesses this. We can now fix, for each n, a family $A_n =$

$\{a_{n,1},\ldots,a_{n,n}\}$ of pairwise disjoint sets of μ-submeasure $\geq \epsilon$. Note that by Lemma 5 and the continuity of μ, if X is an infinite set of members of $\mathcal{B}$ each of μ-submeasure $\geq \epsilon$ then there is an infinite subset Y of X such that $\bigwedge Y \neq \mathbf{0}$. Fix a family $\{f_\xi : \xi < \omega_1\}$ of functions in $\prod_n \{1,\ldots,n\}$ such that for $\xi \neq \eta$ there is l such that $f_\xi(k) \neq f_\eta(k)$, for all $k \geq l$. We build a tower of infinite subsets of $\mathbb{N}$, $I_0 \supseteq_* I_1 \supseteq_* \cdots \supseteq_* I_\xi \supseteq_* \cdots$, for $\xi < \omega_1$, such that

$$b_\xi = \bigwedge \{a_{n,f_\xi(n)} : n \in I_\xi\} \neq \mathbf{0},$$

for each ξ. At a stage α we do the following. If α is a limit ordinal we first find an infinite set J such that $J \subseteq_* I_\xi$, for all $\xi < \alpha$; if $\alpha = \beta + 1$ let $J = I_\beta$. Now, look at the family $\{a_{n,f_\alpha(n)} : n \in J\}$. By Lemma 5 we can find an infinite $I_\alpha \subseteq_* J$ such that $b_\alpha = \bigwedge \{a_{n,f_\alpha(n)} : n \in I_\alpha\} \neq \mathbf{0}$. Notice that if $\xi \neq \eta$ then b_ξ and b_η are incompatible. Therefore $\{b_\xi : \xi < \omega_1\}$ is an uncountable antichain in $\mathcal{B}$, a contradiction. ∎

For the following corollary we only need a version of the P-ideal dichotomy for ideals on $2^{\aleph_0}$ whose relative consistency with CH was proved by Abraham and Todorcevic [AT] without any large cardinal assumptions. This follows from the fact that every nonatomic complete ccc Boolean algebra contains a nonatomic complete subalgebra of size at most $2^{\aleph_0}$.

COROLLARY 4: *Assume ZFC is consistent. Then so is ZFC + CH + "every nonatomic weakly distributive ccc forcing adds a splitting real".* ∎

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After these results were proved, the author learned in late December 2003 that similar results were obtained independently by Balcar, Jech, and Pazak. Moreover, in January 2004, also working independently, Farah and Zapletal deduced Corollary 3 directly from Solecki's results [Sol]. Using this they also obtained a characterization of meager and null ideals in terms of Fubini type properties.

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